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SOLUTIONS FOR THE LANDAU PROBLEM USING SYMPLECTIC REPRESENTATIONS OF THE GALILEI GROUP

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Symplectic unitary representations for the Galilei group are studied. The formalism is based on the noncommutative structure of the star-product, and using group theory approach as a guide, a consistent physical theory in phase space is constructed. The state of a quantum mechanics system is described by a quasi-probability amplitude that is in association with the Wigner function. As a result, the Schrödinger and Pauli–Schrödinger equations are derived in phase space. As an application, the Landau problem in phase space is studied. This shows how this method of quantum mechanics in phase space is to be brought to the realm of spatial noncommutative theories.

Keywords: Moyal product; phase space; quantum fields; Galilei group; Landau problem; Wigner function.

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1. Introduction

The notion of phase space in quantum mechanics arose in 1932, in a seminal paper by Wigner,[1] motivated by the problem of finding a way to improve the quantum statistical mechanics. Wigner introduced his formalism by using a kind of Fourier transform of the density matrix, $\rho(q, q')$, giving rise to what is nowadays called the Wigner function, $f_W(q, p)$, where (q, p) are the coordinates of a phase space manifold (Γ).[1–4] The Wigner function is identified as a quasi-probability density in the sense that $f_W(q,p)$ is real but not positive definite, and as such cannot be interpreted as a probability. However, the integrals $\rho(q) = \int f_W(q,p)dp$ and $\rho(p) = \int f_W(q,p)dq$ are (true) distribution functions.

In the Wigner function approach, each operator, A, defined in the Hilbert space, $\mathcal{H}$, is associated with a function, $a_W(q,p)$, in Γ. This procedure is precisely specified by a mapping $\Omega_W : A \to a_W(q,p)$, such that, the associative algebra of operators defined in $\mathcal{H}$ turns out to be an algebra in Γ, given by $\Omega_W : AB \to a_W \star b_W$, where the star-product, $\star$, is defined by

$$a_W \star b_W = a_W(q,p) \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] b_W(q,p)\,. \tag{1}$$

The reverse is also true: each function, $a_W(q,p)$, in phase space, Γ, is associated with an operator, A, defined in the Hilbert space, $\mathcal{H}$.[2] The result is a noncommutative structure in Γ, that has been explored in different ways.[2–25] In particular, the Wigner function is established directly from experiments, in some cases, and such a result has a strong physical appeal to bring the Wigner formalism to different fields.[26,27] This is the case for studies of the Wigner function in (spatial) noncommutative theories.[28,29] This development is important since the noncommutative theory has found applications in different areas. Indeed, it has been associated with the behavior of non-Abelian gauge fields, with phenomenological effects in condensed matter physics, including aspects of phase transitions,[30–44] and string theory. In the latter case, Connes, Douglas and Schwarz[45] showed that an M-theory can be equivalent to a supersymmetric Yang–Mills field in a noncommutative torus — a result explored by Seiberg and Witten.[46]

Recently,[47–51] unitary representations of symmetry Lie groups have been developed on a symplectic manifold, exploring the noncommutative nature of the star, or Groenewold–Moyal, product and using the mapping Ω_W.[47–49] The scalar representation of Lorentz group for spin 0 and spin 1/2 leads to, for instance, the Klein–Gordon and Dirac equations in phase space, such that the wave functions are closely associated with the Wigner function.[47,48] This provides a fundamental ingredient for the physical interpretation of the formalism, showing its advantage in relation to other attempts to explore, for instance, the Schrödinger equation in phase space;[13–15] such an association is not evident, a fact that represents a hindrance to reach a physical interpretation. In terms of nonrelativistic quantum mechanics, the proposed formalism has been used to treat a nonlinear oscillator perturbatively, to study the notion of coherent states and to introduce a nonlinear Schrödinger equation from the point of view of phase space. In the present work, we apply this symplectic formalism to find the Wigner function for the Landau problem, bringing this method for noncommutative theories.[52,53] Beyond this theoretical aspect, we have to find new solutions for the Landau problem in phase space.

The presentation is organized as follows. In Sec. 2, the symplectic Hilbert space is introduced and we study representations for the Galilei group, deriving the Schrödinger equation. In Sec. 3, Pauli–Schrödinger equation in phase space is derived. In Sec. 4, the Landau problem is considered in phase space, where both quasi-amplitudes of probabilities and the Wigner function are derived. In Sec. 5, final concluding remarks are presented.

2. Schrödinger Equation in Phase Space

In order to study the unitary representations of Lie groups in phase space, it is essential to consider, in a general way, the notion of phase space from which a Hilbert space is introduced.

Consider an analytical manifold $\mathbb{M}$ where each point is specified by coordinates q. The coordinates of each point in the cotangent-bundle $\Gamma = T^*\mathbb{M}$ are denoted by (q,p). The $2N$-dimensional manifold Γ is equipped with two-form, that is defined by

$$\omega = dq \wedge dp$$

and is called the symplectic form. The operator

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} \tag{2}$$

with the symplectic form leads to the Poisson bracket,

$$\{f,g\} = \omega(f\Lambda, g\Lambda) = f\Lambda g\,,$$

where

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\,,$$

with $f = f(q,p)$ and $g = g(q,p)$. The manifold Γ is then called the phase space, and the set of analytical functions $f(q,p)$ is denoted by $C^\infty(\Gamma)$. The vector fields over Γ are given by

$$X_f = f\Lambda = \frac{\partial f}{\partial q}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial q}\,.$$

The Hilbert space associated with Γ is introduced by a set of complex functions, $\psi(q,p)$, which are square integrable in $C^\infty(\Gamma)$, i.e.

$$\int dp\,dq\,\psi^\dagger(q,p)\psi(q,p) < \infty\,.$$

Then, the functions may be defined as $\psi(q,p) = \langle q,p|\psi\rangle$, with

$$\int dp\,dq|q,p\rangle\langle q,p| = 1$$

and

$$\langle\psi|\phi\rangle = \int dp\,dq\,\psi^\dagger(q,p)\phi(q,p)\,,$$

where $\langle\psi|$ is a dual vector of $|\psi\rangle$. This Hilbert space, denoted by $\mathcal{H}(\Gamma)$, is used here as the carrier space for representation of Lie algebras. The set of vectors $|q,p\rangle$ is a basis in $\mathcal{H}(\Gamma)$, such that $\langle q,p|q',p'\rangle = \delta(q-q')\delta(p-p')$.

Consider $\ell = \{a_i, i = 1,2,3,\ldots\}$ a Lie algebra over the (real) field $\mathbb{R}$, of a Lie group $\mathcal{G}$, characterized by the algebraic relations $(a_i, a_j) = C_{ijk}a_k$, where $C_{ijk} \in \mathbb{R}$ are the structure constants and $(\,,)$ is the Lie product. We construct unitary symplectic representations for ℓ, denoted by ℓ_{Sp}, using the star-product, as given in Eq. (1). The associative product in $\mathcal{H}(\Gamma)$ is introduced from Λ, Eq. (2), as a mapping $e^{ia\Lambda} = \star : \Gamma\times\Gamma \to \Gamma$, defined by

$$\begin{aligned}(f\star g)(q,p) &= f(q,p)e^{ia\Lambda}g(q,p)\\ &= \exp\left[ia(\partial_q\partial_{p'} - \partial_p\partial_{q'})\right]f(q,p)g(q',p')\big|_{q',p'=q,p}\,,\end{aligned} \tag{3}$$

where f and g are functions in $C^\infty(\Gamma)$ and $\partial_x = \partial/\partial x$ $(x = p,q)$. The constant a fixes units. The usual associative product is obtained by taking $a = 0$. In addition, to each function, say $f(q,p)$, we introduce an operator in the form $\hat{f} = f(q,p)\star$. Such an operator will be used as the generator of unitary transformations.

In order to consider the nonrelativistic quantum mechanics in phase space, we study representations of the Galilei group in $\mathcal{H}(\Gamma)$. This procedure leads us to the Schrödinger equation in phase space, in a close connection with the Wigner function formalism. Following the standard procedure,[54,55] we construct a unitary representation for the Galilei Lie algebra that is given by following set of commutation relations:

$$\begin{aligned}&[\hat{L}_i,\hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k\,, \quad [\hat{L}_i,\hat{K}_j] = i\hbar\epsilon_{ijk}\hat{K}_k\,,\\ &[\hat{L}_i,\hat{P}_j] = i\hbar\epsilon_{ijk}\hat{P}_k\,, \quad [\hat{K}_i,\hat{P}_j] = i\hbar m\delta_{ij}\mathbf{1}\,, \quad [\hat{K}_i,\hat{H}] = i\hbar\hat{P}_i\,,\end{aligned}$$

with all other relations being null. This is the Lie algebra for the Galilean symmetry with a central extension characterized by m. The various operators defining the Galilei symmetry $\hat{P}$, $\hat{K}$, $\hat{L}$ and $\hat{H}$ are then generators of translations, boost, rotations and time translations, respectively. We represent these operators by using star-operators in the form $\hat{f} = f(q,p)\star$. Then, we consider the Hermitian operators

$$\hat{Q} = q\star = q + \frac{i\hbar}{2}\partial_p\,, \tag{4}$$

$$\hat{P} = p\star = p - \frac{i\hbar}{2}\partial_q\,, \tag{5}$$

such that

$$\hat{K} = m\hat{Q}_i - t\hat{P}_i\,, \tag{6}$$

$$\hat{L}_i = \epsilon_{ijk}\hat{Q}_j\hat{P}_k = \epsilon_{ijk}q_jp_k - \frac{i\hbar}{2}\epsilon_{ijk}q_j\frac{\partial}{\partial p_k} + \frac{i\hbar}{2}\epsilon_{ijk}p_k\frac{\partial}{\partial q_j} + \frac{\hbar^2}{4}\frac{\partial^2}{\partial q_j\partial p_k} \tag{7}$$

and

$$\hat{H} = \frac{\hat{P}^2}{2m} = \frac{1}{2m}\left(\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2\right)$$
$$= \frac{1}{2m}\left[\left(p_1 - \frac{i\hbar}{2}\frac{\partial}{\partial q_1}\right)^2 + \left(p_2 - \frac{i\hbar}{2}\frac{\partial}{\partial q_2}\right)^2 + \left(p_3 - \frac{i\hbar}{2}\frac{\partial}{\partial q_3}\right)^2\right]. \tag{8}$$

The physical content of this representation is derived by observing that $\hat{Q}$ and $\hat{P}$ are transformed by the boost as

$$\exp\left(-i\mathbf{v}\cdot\frac{\hat{K}}{\hbar}\right)\hat{P}_j \exp\left(i\mathbf{v}\cdot\frac{\hat{K}}{\hbar}\right) = \hat{P}_j + mv_j\mathbf{1}\,, \tag{9}$$

$$\exp\left(-i\mathbf{v}\cdot\frac{\hat{K}}{\hbar}\right)\hat{Q}_j \exp\left(i\mathbf{v}\cdot\frac{\hat{K}}{\hbar}\right) = \hat{Q}_j + v_j t\mathbf{1}\,. \tag{10}$$

Furthermore, we have $[\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}\mathbf{1}$. Then, $\hat{Q}$ and $\hat{P}$ can be taken to be the physical observables of position and momentum, respectively. The Galilei boost transforms them according to Eqs. (9) and (10). To be consistent, generators $\hat{L}$ are interpreted as the angular momentum operator and $\hat{H}$ is taken as the Hamiltonian operator. The Casimir invariants of the Lie algebra are given by $I_1 = \hat{H} - \frac{\hat{P}^2}{2m}$ and $I_2 = \hat{L} - \frac{1}{m}\hat{K}\times\hat{P}$, where I_1 describes the Hamiltonian of a free particle and I_2 is associated with the spin degrees of freedom. First, we study the scalar representation; i.e. spin 0.

It would be noted that we have, as usual, other operators in the Hilbert space $\mathcal{H}(\Gamma)$ but without the physical content of observables. This is the case of the c-number operators $\bar{Q} = q\mathbf{1}$ and $\bar{P} = p\mathbf{1}$. Indeed, under the boost, $\bar{Q}$ and $\bar{P}$ transform as,

$$\exp\left(-iv\frac{\hat{K}}{\hbar}\right)2\bar{Q}\exp\left(iv\frac{\hat{K}}{\hbar}\right) = 2\bar{Q} + vt\mathbf{1}$$

and

$$\exp\left(-iv\frac{\hat{K}}{\hbar}\right)2\bar{P}\exp\left(iv\frac{\hat{K}}{\hbar}\right) = 2\bar{P} + mv\mathbf{1}\,.$$

This shows that $\bar{Q}$ and $\bar{P}$ transform as position and momentum variables, respectively. In addition, these operators satisfy $[\bar{Q}, \bar{P}] = 0$. Then $\bar{Q}$ and $\bar{P}$ cannot be interpreted as observables. Nevertheless, they can be used to construct a frame in Hilbert space with the content of phase space. Then, we use the orthogonal basis in $\mathcal{H}(\Gamma)$, given by the vectors $|q,p\rangle$, such that $\bar{Q}|q,p\rangle = q|q,p\rangle$ and $\bar{P}|q,p\rangle = p|q,p\rangle$. It is worth noting that the wave function $\psi(q,p,t) = \langle q,p|\psi(t)\rangle$ is associated with the state of the system, but its time evolution and physical content remain to be specified.

In this sense, $\psi(q,p)$ is a wave function but not with content of the usual quantum mechanics state, for q and p are eigenvalues of the operators $\bar{Q}$ and $\bar{P}$ which are ancillary variables (not observables).

The time evolution equation for $\psi(q,p,t)$ is derived by using the generator of time translations, such that

$$\psi(t) = e^{\frac{-i\hat{H}t}{\hbar}}\psi(0)\,. \tag{11}$$

This leads to

$$i\hbar\partial_t\psi(q,p;t) = \hat{H}(q,p)\psi(q,p;t)\,,$$

where $\hat{H}(q,p) = H(q,p)\star$. This is the Schrödinger equation represented in phase space.[47]

The average of a physical observable $\hat{A}(q,p) = a(q,p;t)\star$, in the state $\psi(q,p)$ is given by

$$\begin{aligned}\langle A\rangle &= \int dq\,dp\,\psi^\dagger(q,p)\hat{A}(q,p)\psi(q,p)\\ &= \int dq\,dp\,a(q,p)[\psi(q,p)\star\psi^\dagger(q,p)]\,.\end{aligned} \tag{12}$$

The association of $\psi(q,p,t)$ with the Wigner function is,[47]

$$f_W(q,p) = \psi(q,p,t)\star\psi^\dagger(q,p,t)\,. \tag{13}$$

Indeed, this function satisfies the Liouville–von Neumann equation,[47] and the probability density in configuration space is

$$\rho(q) = \int dp[\psi(q,p)\star\psi^\dagger(q,p)] = \int dp\,\psi(q,p)\psi^\dagger(q,p)\,, \tag{14}$$

while, in momentum space, it is

$$\rho(p) = \int dq[\psi(q,p)\star\psi^\dagger(q,p)] = \int dq\,\psi(q,p)\psi^\dagger(q,p)\,. \tag{15}$$

It is important to emphasize that the average of an observable is consistent with the Wigner formalism, i.e. from Eqs. (12) and (13), we have

$$\langle A\rangle = \int dq\,dp\,a(q,p)f_W(q,p;t)\,.$$

This provides a complete set of physical rules to interpret representations and opens the way to study other improvements. For instance, considering a system in an external magnetic field, a gauge transformation is introduced by following the usual procedure: $\hat{P} \to \hat{P} + ieA(q)$. Another aspect of interest is that the phase space description of quantum mechanics is fully presented in a self consistent way in terms of representations. This is not the case of usual procedures, where we first solve the Schrödinger equation in order to proceed further with the construction of the Wigner function changing the representation in an intricate way.

3. Pauli–Schrödinger Equation in Phase Space

Next, we consider the spin 1/2 representation. In this case, $\mathcal{H}(\Gamma)$ has to be a spinorial Hilbert space. This leads to the following Pauli–Schrödinger equation in phase space, describing an electron in a magnetic field, i.e.

$$i\hbar\partial_t\psi(q,p;t) = H(q,p)\star\psi(q,p;t)\,,$$

where $\psi(q,p;t)$ is a two-dimensional spinor in phase space

$$\psi(q,p;t) = \begin{pmatrix}\phi(q,p;t)\\ \xi(q,p;t)\end{pmatrix}$$

and

$$H(q,p)\star = \frac{1}{2m}(\hat{P}+ie\hat{A})^2 + \mu_B\hat{L}\cdot\sigma\,,$$

with $\hat{L}$ being given by Eq. (7) and $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ are Pauli matrices. The external field $\hat{A}$ is such that $\hat{A} = A(q)\star$. In the next section, we use this formalism and the gauge transformation to develop a symplectic representation for (spatial) noncommutative theories, addressing the Landau problem in phase space.

4. Landau Problem in Phase Space

The Landau problem[56,57] refers to a study of the behavior of an electron moving in a uniform external magnetic field, orthogonal to the plane containing the electron. The description can be considered as a system with a noncommuting spatial coordinates. The objective is to determine the Wigner function for the electron, following the symplectic approach, to compare it with standard procedures.[52]

The Hamiltonian with noncommutative position coordinates may be introduced by using the star product defined by

$$\star_\theta = \exp\frac{i\theta}{2}\left(\overleftarrow{\partial_x}\,\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\,\overrightarrow{\partial_x}\right).$$

Then, we take space coordinates to obey the Moyal brackets,

$$[x,y]_\theta = x\star_\theta y - y\star_\theta x = i\theta\,,$$

where θ is a constant parameter.

The quantization of the system is considered by usual commutation relations,

$$[q_i,p_j] = i\hbar\delta_{ij}\,,$$

where $q = (q_i) = (x,y)$ and $p = (p_i) = (p_x,p_y)$. The derivatives in q are the momentum operator, $p = -i\hbar\partial_q$.

The noncommutative Hamiltonian[52] is written as

$$H = \frac{1}{2}\left[\left((1+k)p_x - \frac{B}{2}y\right)^2 + \left((1+k)p_y - \frac{B}{2}x\right)^2\right], \tag{16}$$

where B is the intensity of the magnetic field. In this problem, we have θ given by: $\theta = \theta_{12}$, where $\theta_{ij} = 2\varepsilon_{ij}/B$ with ε_{ij} being the antisymmetric tensor, and $k = \theta B/4 = 1/2$.

Then, the stationary Schrödinger equation in phase space for the Landau problem is

$$\frac{1}{2}\left[\left((1+k)p_x - \frac{B}{2}y\right)^2 + \left((1+k)p_y - \frac{B}{2}x\right)^2\right] \star \psi(q,p) = E\psi(q,p)\,. \tag{17}$$

This equation is solved with an algebraic approach, using the star-operators.

Let us define,

$$\hat{a} = a\star = \frac{1}{\sqrt{2B(1+k)}}\left[\left(p_x \star - \frac{B}{2}y\star\right) - i\left(p_y \star + \frac{B}{2}x\star\right)\right] \tag{18}$$

and

$$\hat{a}^\dagger = a^\dagger\star = \frac{1}{\sqrt{2B(1+k)}}\left[\left(p_x \star - \frac{B}{2}y\star\right) + i\left(p_y \star + \frac{B}{2}x\star\right)\right], \tag{19}$$

such that $[a\star, a^\dagger\star] = 1$. Then, $a\star$ and $a^\dagger\star$ are the annihilation and creation operators, respectively, in a basis defined in phase space. In addition,

$$[H\star, a\star a^\dagger\star] = [H\star, a^\dagger \star a\star] = 0\,.$$

Then, Eq. (17) is written as

$$(1+k)B\left(a^\dagger \star a \star + \frac{1}{2}\right)\psi(q,p) = E\psi(q,p)\,. \tag{20}$$

The eigenvalues of $H\star$ are derived by using,

$$a^\dagger \star a \star \psi_n(q,p) = n\psi_n(q,p)\,.$$

Then,

$$E_n = (1+k)B\left(n + \frac{1}{2}\right).$$

For the ground state ψ_0, such that $a \star \psi_0 = 0$, we have

$$\begin{aligned}(1+k)&(p_x + p_y)\psi_0(q,p) \\ &= \left[\frac{B}{2}(x+y) - \frac{1}{2}(1+k)(\partial_x + \partial_y) + \frac{B}{4}(\partial_{p_x} + \partial_{p_y})\psi_0(q,p)\right].\end{aligned} \tag{21}$$

A solution for this equation is

$$\psi_0(q,p) = N_0 e^{\frac{-1}{B(1+k)}\left[\left((1+k)p_x - \frac{B}{2}y\right)^2 + \left((1+k)p_y + \frac{B}{2}x\right)^2\right]} = N_0 e^{\frac{-2H}{B(1+k)}}\,,$$

where N_0 is a normalization constant. Using $\int dq\, dp\, \psi_0^\dagger \star \psi_0 = 1$, we have

$$\psi_0(q,p) = \sqrt{\frac{e}{\pi}} e^{\frac{-2H}{(1+k)B}}\,.$$

From the ground state, we obtain $\psi_n(q,p)$ by

$$\psi_n(q,p) = \frac{1}{\sqrt{n!}}(a^\dagger \star)^n \psi_0\,,$$

that results in

$$\psi_n(q,p) \sim \left[\left((1+k)p_x - \frac{B}{2}y\right)^2 - i\left((1+k)p_y + \frac{B}{2}x\right)^2\right]^n e^{\frac{-2h}{(1+k)B}}\,.$$

We derive the Wigner function associated with each $\psi_n(q,p)$ by using Eq. (13), i.e.

$$f_W^n(q,p) \sim (a^\dagger)^n \star e^{\frac{-2H}{(1+k)B}} \star a^n\,.$$

In particular, for $n = 1, 2$, we find

$$f_W^{(1)}(q,p) \sim \left[1 - \frac{4H}{(1+k)B}\right] e^{\frac{-2H}{(1+k)B}}\,,$$

$$f_W^{(2)}(q,p) \sim \left[1 - 4\frac{4H}{(1+k)B} + \left(\frac{4H}{(1+k)B}\right)^2\right] e^{\frac{-2H}{(1+k)B}}\,.$$

For an arbitrary n, we have

$$f_W^{(n)}(q,p) \sim L_n\left(\frac{4H}{(1+k)B}\right) e^{\frac{-2H}{(1+k)B}}\,, \tag{22}$$

where $L_n(x)$ are the Laguerre polynomials. This result for the Landau problem in phase space is the same as the one derived by Dayi and Kelleyane,[52] following a different method.

In Eq. (22), we realize that the Wigner function corresponding to the Landau problem depends on the parameter θ, since $k = \frac{\theta B}{4}$. However, from a theoretical point of view, one of the main problems in noncommutative models is the determination of the parameter θ. In most models, this parameter is arbitrary. A question of great relevance to the acceptance of noncommutative models as candidates for the description of physical phenomena is how the parameter θ may be related to the observable physical quantities.[58–60]

5. Concluding Remarks

In this work, we have developed symplectic representations of the Galilei group, that give rise to quantum theories in phase space. The Schrödinger and Pauli–Schrödinger equations are derived, describing particles of spin 0 and spin 1/2, respectively. As an application, we have considered the Landau problem in phase space, and the Wigner function has been derived. The symplectic representations are constructed by using the notion of the star-product, as a noncommutative geometrical ingredient. Then, a Hilbert space is defined from a manifold with characteristics of phase space. The states are represented by wave functions, that are interpreted as quasi-probability amplitudes. This aspect gives rise to a connection with the Wigner function. We have recovered solutions in the form of the Wigner function, derived by following the standard method based on the density matrix. Here, however, we have found other solutions. Indeed, it is important to emphasize that those new solutions are linear superposition of the quasi-amplitudes of probabilities. In addition, the analysis of the Landau problem, as a noncommutative formalism, shows a method to study noncommutative theories in phase space.

As a final observation, it is worth noting that the group theoretical approach gives us strict directions to develop the phase space method based on unitary representations with a physically consistent interpretation. This is true for the kinematical symmetry as the Galilei and Poincaré group, and provides a direct way to derive the Wigner function for gauge theories, as it was shown here with the derivation of the Pauli–Schrödinger in phase space. The complete analysis of gauge theory in phase space will be presented in another place.

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